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Comment on "A Prediction of Particle Behavior via the Basset-Boussinesq-Oseen Equation"

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Introduction

In a recent Note, Catalano[1] has given an analytical solution of the Basset-Boussinesq-Oseen (BBO) equation. This solution was applied to the interesting problem of the discrepancy between theory and laser velocimetry applications (LDA) measurements in the case of unstationary flows arising especially in Raleigh-Bénard convection or at the onset of turbulence in a circular Couette flow. This discrepancy is observed mainly for high frequencies. In the following we would like to comment on the author's assumptions: Accepting, as Catalano did, the BBO equation in the form given by Tchen[2] (for a thorough discussion of this topic, we refer to Reek's recent paper),[5] we do not agree that the Basset term can be neglected. For example, we look at the experiments cited by Catalano[6,7]: Taking into account the properties of liquid helium, it turns out that Hinze's[4] conditions for the neglect of the Basset term are not fulfilled for marker particles with a diameter less 10 μm. Also, it should be mentioned that Catalano neglected the mass addition term, which represents the force to accelerate the virtual mass of the particle relative to the fluid.

In extending Catalano's analytical investigation, we present a solution of the complete BBO equation. We use a standard method[3] for solving the inhomogeneous Volterra-integral equation of second kind. Our solution accounts for the Basset term describing the time behavior of the flow in the neighborhood of the particles. We proceeded in this way because we believe that, in unstationary flows of the type considered, near the onset of turbulence, the Bassset term cannot be neglected. The resultant closed-form solution for the particle motion should be examined for the case of Rayleigh-Bénard flow as well. The nomenclature we use is the same as Catalano's. At this opportunity we also want to make some corrections of Eqs. (2), (3), (6), (7), (15), (17), (18), (20), and (21) of Ref. 1. Further, we would like to mention that Catalano did not remark that the lift forces [see Eqs. (4) and (5)] are directed perpendicular to the relative velocity vector $u_p - u_F$. So the inclusion of the lift force is rather questionable in the obviously one-dimensional case presented.[1]

Governing Equation

The Lagrangian equation of motion of a spherical particle suspended in a viscous fluid is the so-called Basset-Boussinesq-Oseen (BBO) equation. A form following from Tchen's proposal[2] is

$$a_p + \int_{\tau_0}^{\tau} a_p \left(A_1 + \frac{C}{(\tau - \tau')^{1/2}} \right) \mathrm{d}\tau'$$
$$= Ba_F + \int_{\tau_0}^{\tau} a_F \left(A_1 + \frac{C}{(\tau - \tau')^{1/2}} \right) \mathrm{d}\tau' + D \tag{1}$$

with the particle acceleration $a_p = a_p(\tau) = \mathrm{d}u_p/\mathrm{d}\tau$ and the fluid acceleration $a_F = a_F(\tau) = \mathrm{d}u_F/\mathrm{d}\tau$. One of the conditions of its validity is that the marking particle remains in its own eddy and that Stokes' linear resistance law holds. The coefficient A_1 is the same as in Ref. 1 and should also include the effect of the lift forces due to particle rotation and to a velocity gradient.

Defining

$$\tau^* = A_1 \tau, \qquad D_1 = D/A_1, \qquad C_1 = C/(A_1)^{1/2} \tag{2}$$

we get from Eq. (1)

$$a_p + \int_{\tau_0}^{\tau^*} (a_p - a_F) \left(1 + \frac{C_1}{(\tau^* - \tau')^{1/2}} \right) \mathrm{d}\tau' = Ba_F + D_1 \tag{3}$$

We subsume all terms concerning the motion of the fluid in $n(t^*)$ and get the following inhomogeneous Volterra integral equation:

$$a_p(\tau^*) + \int_{\tau_0}^{\tau^*} a_p(\tau') K(\tau^* - \tau') \mathrm{d}\tau' = n(\tau^*) \tag{4a}$$

where the second term of the kernel has Abel-type form

$$K(\tau^* - \tau') = 1 + C_1/(\tau^* - \tau')^{1/2} \tag{4b}$$

It can be shown that the resolvent of the kernel K or Eq. (4) depends only on the difference $\tau - \tau'$. We call it $r(\tau^* - \tau')$, and we can write the solution of Eq. (4) in the form

$$a_p(\tau^*) = n(\tau^*) - \int_{\tau_0}^{\tau^*} r(\tau^* - \tau') n(\tau') \mathrm{d}\tau' \tag{5}$$

Using the Laplace transformation with

$$\mathcal{L}(a_p) = A_p(s) \qquad \mathcal{L}(r) = R(s)$$
$$\mathcal{L}(n) = N(s) \quad \text{and} \quad \mathcal{L}(K) = L(s)$$

and the inversion equation

$$r(\tau^*) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} R(s) \exp(s\tau^*) \mathrm{d}s \tag{6a}$$

where

$$R(s) = L(s)/[1 + L(s)] \tag{6b}$$

and σ is a sufficiently large real number.

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We can easily determine the solution of Eq. (4). We get for the resolvent:

$$R(s) = [C_1(\pi s)^{1/2} + 1][s + C_1(\pi s)^{1/2} + 1]$$

Applying Jordan's Lemma[3] one can show that the integral in Eq. (6) is equal to the sum of the residuals (Res) of the integrand. Thus we get

$$u_p(\tau^*) = \int_{\tau_0}^{\tau^*} \Big\{ n(t) - \int_{\tau_0}^{t} \sum_{i=1}^{2} \mathrm{Res}[R(s_i)\exp(s_i x)] \Big|_{x=t-\tau'} n(\tau')\mathrm{d}\tau' \Big\} \mathrm{d}t \qquad (7)$$

By the method of decomposition in partial fractions, one can show that the denominator polynomial is of second degree. It has the form

$$f(s) = s^2 + (2 - C_1^2\pi)s + 1 \qquad (8)$$

It follows from an order-of-magnitude estimation of the coefficients that the two roots are conjugated complex. Their real parts lie in the left half-space of the complex plane:

$$s_{1,2} = (C_1^2\pi/2) - 1 \pm i[1 - (2 - C_1^2\pi)^2/4]^{1/2} \qquad (8a)$$

We note that, for large particle-density ratio ($\sigma \gg 1$) and negligible lift forces, $C_1 \cong 3/2.\ (2/\pi\sigma)^{1/2}$, while for $\sigma \approx 1$, $C_1 = (2/\pi)^{1/2}$. In the latter case, we get asymptotic stability:

$$s_{1,2} = \pm i \qquad (8b)$$

One can recognize that, generally, the Basset term in the BBO equation gives rise to damped oscillations of the particle. However, with increasing particle rotation velocity and spatial gradients of the mean flow, velocity C_1 rises and the Basset term leads to unstable modes. This happens also for absent lift force when $2A_1$ is less than $81/(\sigma + 0{,}5)^2$. Applying the Laplace-transformation directly to Eq. (3), we get

$$U_p(s) = \frac{U_F(s)[Bs + C_1(\pi s)^{1/2} + 1] + D_1 s}{s[s + C_1(\pi s)^{1/2} + 1]} \qquad (9)$$

with

$$\mathcal{L}(a_p) = A_p(s),\ \mathcal{L}(r) = R(s),\ \mathcal{L}(n) = N(s),\ \mathcal{L}(K) = L(s)$$

Conclusions

From Eq. (9) the amplitude and phase ratio can be determined immediately by complex calculations setting $s = j\omega$. The results are presented in Hinze,[4] using a Fourier integral representation of $u_p(t)$ and $u_F(t)$. It can easily be shown that the determinant of the antimetric transfer matrix of the amplitudes of particle and fluid motion is equal to the ratio of the Lagrangian energy-spectrum functions. For short diffusion times, the main contribution to the diffusion coefficient comes from the high-frequency components of motion. The smaller the marker-particles, the shorter the particle relaxation time. This indicates to what extent the particle follows the high frequency fluctuations of the fluid.

Corrections Concerning Ref. 1

The equations labeled as in Ref. 1 read correctly as follows:

Eq. (2): $\dfrac{\mathrm{d}U_p}{\mathrm{d}\tau} + AU_p = B\dfrac{\mathrm{d}U_F}{\mathrm{d}\tau} + AU_F + D$

In Eqs. (3), (6), and (7): the dynamic viscosity μ_F in the coefficient $D = d^2g/\mu_F$ should be substituted by the kinematic viscosity ν_F.

Eq. (15): $f \simeq n_1 f_1 + n_2 f_2 + \ldots + n_N f_N$

Eq. (17): $U_{pj}(\tau^*) = \tilde{A}_j[(1/\tilde{\omega}_j^* - B)\sqrt{2}]\cos(\tilde{\omega}_j^*\tau^* + \Theta_j') + D_1$

Eq. (18): $\dfrac{U_{pj}}{U_{Fj}} = \dfrac{1/\omega_j^* - B}{(\sqrt{2})^{-1}} - \dfrac{D_1}{\tilde{A}_j}\mathrm{sgn}(1/\omega_j^* - B)$

In Eq. (20): $\mathrm{sgn}\left(\dfrac{1}{\omega_j^* - B}\right)$ must be subtituted by

$\mathrm{sgn}\left(\dfrac{1}{\omega_j^*} - B\right)$

In Eq. (21): the factor $\dfrac{\sigma d g^2}{18\mu_F}$ by $\dfrac{\sigma d^2 g}{18\mu_F}$

References

[1]Catalano, G. D., "A Prediction of Particle Behavior via the Basset-Boussinesq-Oseen Equation," *AIAA Journal*, Vol. 23, Oct. 1985, pp. 1627-1628.

[2]Tchen, C. M., Dissertation, University of Delft, the Netherlands, 1947.

[3]Smirnow, W. I., "Lehrgang der Höheren Mathematik," VEB Deutscher Verlag des Wissenschaften, Berlin, 1964, Part III/2, Chap. 60, p. 189.

[4]Hinze, J. O., *Turbulence*, 2d ed., McGraw-Hill Book Company, New York, 1975, pp. 460-471.

[5]Reeks, M. W. and McKee, S., "The Dispersive Effect of Basset History Forces on Particle Motion in Turbulent Flow," *Physics of Fluids*, Vol. 27, 1984, pp. 1573-1582.

[6]Gorman, M., Reith, L. A., and Swinney, H. L., "Phenomena in Circular Couette Flow," *Annals of New York Academy of Sciences*, Vol. 357, 1980, pp. 10-21.

[7]Gollub, J. P., Binson, S. V., and Stunman, J., "On the Subharmonic Route to Turbulent Correction," *Annals of New York Academy of Sciences*, Vol. 357, 1980, pp. 22-27.

Reply by Author to F. Ebert and S. U. Schöffel

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THE author wishes to express his appreciation to Ebert and Schöffel for their observations of typographical errors in equations described in a previous article.[1] These corrections certainly enhance the usefulness of the results. In addition, the author wishes to comment on the assumptions used in the development.

Consider the Basset-Boussinesq-Oseen equation depicted in Eq. (1) in the author's article. The form of the equation suggests the consideration of the relative magnitudes of the different terms and a resultant simplification. For the case when the density of the flow tracing particle is substantially greater than the density of the fluid, as is commonly the case in laser velocimetry (LV) applications, it is possible to neglect the effect of the added mass and to set the added mass coefficient equal to zero.[2] Also, again for the special case of interest to

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